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Inclusion Principle for the Rayleigh-Ritz Based Substructure Synthesis

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This paper is concerned with the convergence characteristics of a Rayleigh-Ritz based substructure synthesis developed earlier by these authors. According to this substructure synthesis, the motion of every substructure is modeled in terms of quasicomparison functions, which are linear combinations of admissible functions capable of satisfying all the boundary conditions. A consistent kinematical procedure permits the aggregation of the various substructures and it ensures compatibility without the need of imposing constraints. The substructure synthesis is characterized by the fact that the mass and stiffness matrices of the model possess the embedding property, which implies that the matrices defining the $(n+1)$-order approximation are obtained by simply adding one row and one column to the matrices defining the n-order approximation. As a result, the computed eigenvalues satisfy the inclusion principle, which in turn can be used to demonstrate uniform convergence of the approximate solution. A numerical example illustrates the inclusion principle.

I. Introduction

MANY structures, such as fixed-wing aircraft, helicopters, space stations, flexible robots, etc., can be modeled as assemblages of interacting flexible substructures. A problem arising frequently in such structures is how to produce a relatively accurate model with as few degrees of freedom as possible. Due to the nature of the problem, a model derived by a substructure synthesis method seems to be indicated. The idea of substructure synthesis is to model the various substructures independently and then to impose certain compatibility conditions to force the individual substructures to act as a single structure.

A method proposed by Hurty[1,2] in the early 1960s, and known as component-mode synthesis, consists of modeling the motion of the individual substructures, referred to as components, by means of "component modes." He uses three types of modes to represent the motion of a substructure: rigid-body modes, constraint modes, and normal modes. An approach by Craig and Bampton[3] differs from that of Hurty[1,2] mainly in the selection of the component modes. The substructures modeled individually in Refs. 1–3 are made to act together as a single structure by eliminating the redundant generalized coordinates arising from the fact that displacements at points common to two adjacent substructures are included twice in the overall problem formulation, once for each substructure. The elimination process is based on the use of constraint equations resulting from the enforcement of compatibility conditions, which amounts to saying that the displacement of a boundary point shared by two substructures is the same. In another approach to the problem, proposed by Benfield and Hruda[4] the effect of adjacent substructures is taken into account by subjecting a given substructure to inertial and stiffness loadings at the boundaries.

In the late 1970s and early 1980s Meirovitch and Hale[5,6] demonstrated that the component-mode synthesis and all its variants are essentially different forms of the Rayleigh-Ritz method. Consistent with this, an approximate solution can be constructed from the space of admissible functions,[7] i.e., the functions need not be component modes. Indeed, component modes merely represent subspaces of the much larger space of admissible functions, and component-mode synthesis is part of a larger picture. Hence, it is more appropriate to refer to the approach as substructure synthesis.[5,6]

In the classical Rayleigh-Ritz method, a sequence of approximating solutions is constructed from the space of either admissible functions or comparison functions,[7] depending on the problem formulation, where admissible functions need satisfy only the geometric boundary conditions and comparison functions must satisfy all the boundary conditions. However, if the problem is formulated by a variational approach, admissible functions suffice. In the case of flexible multibody systems, boundary conditions for a given substructure cannot be defined independently of the adjacent substructures, so that the use of comparison functions is not an option. Hence, the only alternative is to use mere admissible functions, which include the various "substructure modes" as special cases. The use of mere admissible functions, however, raises serious questions as to the speed of convergence, which in turn is likely to require models with relatively large numbers of degrees of freedom.

Quite recently, Meirovitch and Kwak[8] demonstrated that the common use of the classical Rayleigh-Ritz method has an implicit flaw that can impair its convergence characteristics. As pointed out above, if the eigenvalue problem is formulated as a variational problem, the Rayleigh-Ritz method consists of constructing a sequence of approximate solutions from the space of admissible functions. These admissible functions are commonly taken as members of the same family of functions. In many cases, solutions in the form of series of admissible functions of the same type are characterized by poor convergence, which can be traced to the fact that a finite linear combination of admissible functions of the same type is not able to satisfy the natural boundary conditions, in addition to not satisfying the differential equation. To correct this flaw, Meirovitch and Kwak[8] proposed that the approximating sequence be from the space of quasicomparison functions, instead of merely from the space of admissible functions. The quasicomparison functions represent a new class of functions with superior convergence characteristics. They are defined as linear combinations of admissible functions capable of sat-

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isfying all the boundary conditions of the problem. This implies that the admissible functions must be of several types.

The concepts developed in Ref. 8 for single elastic members have been extended to substructure synthesis in Ref. 9. Indeed, it is in substructure synthesis that the power and versatility of quasicomparison functions become evident. Reference 9 presents a substructure synthesis theory identified closely with the Rayleigh-Ritz theory and based on the concept of quasicomparison functions.

One question that arises frequently in connection with approximate solutions of differential eigenvalue problems is that of convergence. In the case of the classical Rayleigh-Ritz method, the question is largely answered by the inclusion principle, according to which the eigenvalues computed on the basis of an $(n+1)$-order approximate model bracket those of an n-order model. The proof of the inclusion principle relies on the embedding property of the mass and stiffness matrices, which simply implies that the addition of one term to the series approximating the solution results in the addition of one row and one column to these matrices, without affecting the previously computed entries. The inclusion principle can be used to demonstrate that the computed eigenvalues converge to the actual eigenvalues, where the convergence is uniform and from above.

This paper demonstrates that the Rayleigh-Ritz based substructure synthesis of Ref. 9 has a distinct advantage over the other procedures, because it permits mathematical proof of convergence. This proof is based on the embedding property of the mass and stiffness matrices, which in turn guarantees the existence of the inclusion principle.

A numerical example demonstrates the validity of the inclusion principle for the Rayleigh-Ritz based substructure synthesis as well as the convergence characteristics of the eigenvalues computed on the basis of such models.

II. Inclusion Principle for Positive Definite Real Symmetric Matrices

Let us consider the eigenvalue problem associated with a positive definite real symmetric matrix A and write it in the form

$$Ax = \lambda x \tag{1}$$

If A is an $(n+1) \times (n+1)$ matrix, then there are $n+1$ eigenvalues, which can be arranged in ascending order, so that $\lambda_1 \leq \lambda_2 \leq, \ldots, \leq \lambda_{n+1}$. Next, we consider the eigenvalue problem associated with the $n \times n$ positive definite real symmetric matrix B, or

$$By = \gamma y \tag{2}$$

where the n eigenvalues of B are arranged so as to satisfy $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$. Next, we assume that B is embedded in A, which implies that B is obtained from A by striking out one row and the corresponding column. In practical situations it is the last row and column stricken out, so that the embedding property can be expressed as

$$A = \begin{bmatrix} & B & & x \\ & & & x \\ x & & x & x \end{bmatrix} \tag{3}$$

Then, it is shown in Ref. 7 that, if any two matrices A and B possess an embedding property of the type exemplified by Eq. (3), the eigenvalues $\lambda_r (r = 1, 2, \ldots, n+1)$ of A and the eigenvalue $\gamma_r (r = 1, 2, \ldots, n)$ of B satisfy the inequalities

$$\lambda_1 \leq \gamma_1 \leq \lambda_2 \leq \gamma_2 \leq \ldots \leq \lambda_n \leq \gamma_n \leq \lambda_{n+1} \tag{4}$$

or, the eigenvalues of A bracket the eigenvalues of B. Inequalities in Eq. (4) represent *the inclusion principle*.[7] The principle is fundamental to the Rayleigh-Ritz theory.

III. Inclusion Principle for Structural Models Derived by the Raleigh-Ritz Method

The differential eigenvalue problem associated with a distributed structure can be expressed in the form[7]:

$$\mathcal{L}w(P) = \lambda m(P) w(P), \; P \in D \tag{5}$$

where $w(P)$ is the displacement at the nominal point P of the structure, $\mathcal{L}$ is a self-adjoint positive definite differential operator of order $2p$, reflecting the stiffness of the structure, $m(P)$ is the mass density and D is the domain of the structure. The displacement is subject to the boundary conditions

$$B_i w(P) = 0, \; i = 1, 2, \ldots, p; \; P \in S \tag{6}$$

where B_i are differential operators of maximum order $2p - 1$ and S represents all the boundary points of D. The solution of the eigenvalue problem described by Eqs. (5) and (6) consists of a denumerably infinite set of real and positive eigenvalues λ_r and associated eigenfunctions $w_r(P) (r = 1, 2, \ldots)$.

An alternative to solving the differential eigenvalue problem, Eqs. (5) and (6), consists of rendering stationary the Rayleigh quotient[7]

$$R(w) = \frac{[w, w]}{(\sqrt{m}w, \sqrt{m}w)} \tag{7}$$

where $[w, w]$ is an energy inner product related to the elastic potential energy and $(\sqrt{m}w, \sqrt{m}w)$ is a weighted inner product related to the kinetic energy. For many problems in structural dynamics, rendering the Rayleigh quotient stationary represents a more attractive problem than solving the differential eigenvalue problem.

Exact solutions to the eigenvalue problem can be obtained in only a few cases, almost invariably involving uniform stiffness and mass distributions. Hence, more often than not, we must be content with an approximate solution. One method permitting an approximate solution to the eigenvalue problem is the Rayleigh-Ritz method. The method consists of assuming an approximate solution in the form of the finite series

$$w^{(n)}(P) = \sum_{i=1}^{n} a_i \phi_i(P) = \boldsymbol{\phi}^T(P) \boldsymbol{a}^{(n)} \tag{8}$$

where a_i are undetermined coefficients and $\phi_i(P)$ are admissible functions[7]; $\boldsymbol{a}^{(n)}$ and $\boldsymbol{\phi}(P)$ are the associated vectors. The admissible functions must be from a complete set, which implies that the approximate solution can be made as accurate as desired by merely increasing the number n of terms in the series. Introducing Eq. (8) into Eq. (7), we obtain

$$R = \frac{\boldsymbol{a}^{(n)T} K^{(n)} \boldsymbol{a}^{(n)}}{\boldsymbol{a}^{(n)T} M^{(n)} \boldsymbol{a}^{(n)}} \tag{9}$$

where

$$K^{(n)} = [\boldsymbol{\phi}(P), \boldsymbol{\phi}(P)] \tag{10a}$$

$$M^{(n)} = (\sqrt{m}\boldsymbol{\phi}(P), \sqrt{m}\boldsymbol{\phi}(P)) \tag{10b}$$

are $n \times n$ positive definite symmetric stiffness and mass matrices. Rendering the Rayleigh quotient, Eq. (9), stationary is equivalent to solving the algebraic eigenvalue problem

$$K^{(n)} \boldsymbol{a}^{(n)} = \lambda^{(n)} M^{(n)} \boldsymbol{a}^{(n)} \tag{11}$$

The solution consists of the computed eigenvalues $\lambda_r^{(n)}$ and eigenvector $\boldsymbol{a}_r^{(n)}$. The computed eigenvalues $\lambda_r^{(n)}$ represent approximations to the actual eigenvalues $\lambda_r (r = 1, 2, \ldots, n)$. On the other hand, inserting the eigenvectors into Eq. (8), we obtain the computed eigenfunctions

$$w_r^{(n)}(P) = \boldsymbol{\phi}^T(P)\boldsymbol{a}_r^{(n)}, \quad r = 1, 2, \ldots, n \tag{12}$$

It follows that the Rayleigh-Ritz method can be interpreted as a modeling procedure whereby an infinite-dimensional distributed structure is approximated by a finite-dimensional discrete model. In essence, the Rayleigh-Ritz method is a discretization and truncation procedure replacing a differential eigenvalue problem by an algebraic eigenvalue problem.

The eigenvalue problem, Eq. (11), can be reduced to standard form, i.e., one in terms of a single matrix. To this end, we recognize that $M^{(n)}$ is real symmetric and positive definite and use the Cholesky decomposition[7] to write

$$M^{(n)} = L^{(n)}L^{(n)T} \tag{13}$$

where $L^{(n)}$ is a lower triangular matrix. Inserting Eq. (13) into Eq. (11), introducing the notation

$$L^{(n)T}\boldsymbol{a}^{(n)} = \boldsymbol{b}^{(n)} \tag{14}$$

and premultiplying both sides of the resulting equation by $(L^{(n)})^{-1}$, we obtain the eigenvalue problem in standard form

$$A^{(n)}\boldsymbol{b}^{(n)} = \lambda^{(n)}\boldsymbol{b}^{(n)} \tag{15}$$

where

$$A^{(n)} = (L^{(n)})^{-1}K^{(n)}(L^{(n)})^{-T} = A^{(n)T} \tag{16}$$

is a positive definite real symmetric matrix, in which $(L^{(n)})^{-T} = (L^{(n)T})^{-1} = [(L^{(n)})^{-1}]^T$.

The accuracy of the results obtained by the Rayleigh-Ritz method can be improved by adding terms to the series (8). Indeed, adding one term to the series and following the established procedure, we obtain the eigenvalue problem[10]

$$A^{(n+1)}\boldsymbol{b}^{(n+1)} = \lambda^{(n+1)}\boldsymbol{b}^{(n+1)} \tag{17}$$

where

$$A^{(n+1)} = \begin{bmatrix} A^{(n)} & \boldsymbol{p} \\ \boldsymbol{p}^T & q \end{bmatrix} \tag{18}$$

in which $\boldsymbol{p}$ is an n vector and q is a scalar. Hence, matrix $A^{(n)}$ is embedded in the matrix $A^{(n+1)}$. It follows that *the inclusion principle applies to discrete models derived by the Rayleigh-Ritz method*, or

$$\lambda_1^{(n+1)} \le \lambda_1^{(n)} \le \lambda_2^{(n+1)} \le \lambda_2^{(n)} \le \ldots \le \lambda_n^{(n+1)} \le \lambda_n^{(n)} \le \lambda_{n+1}^{(n+1)} \tag{19}$$

The inclusion principle permits us to state that the approximate eigenvalues computed by the Rayleigh-Ritz method converge to the actual eigenvalues, or

$$\lim_{n\to\infty} \lambda_r^{(n)} = \lambda_r, \quad r = 1, 2, \ldots, n \tag{20}$$

where the convergence is uniform and from above.

IV. Convergence Improvement for the Rayleigh-Ritz Method

The Rayleigh-Ritz theory guarantees convergence of the computed eigenvalues to the actual eigenvalues as the number of terms in series (8) approaches infinity. Whereas this is reassuring, the prospect of solving eigenvalue problems of very large order is not comforting. This prospect is quite real in problems involving natural boundary conditions.[8] In such cases, the choice of admissible functions can be critical.

In solving the eigenvalue problem by rendering Rayleigh's quotient stationary, the approximate solution has the form of a linear combination of admissible functions. We recall that admissible functions are p times differentiable and satisfy the geometric boundary conditions of the problem.[7] As demonstrated in Ref. 8, the fact that admissible functions do not satisfy the natural boundary conditions can lead to serious convergence problems. This idea can be best demonstrated by means of a specific example. To this end, we consider a cantilever beam supported by a spring at the right end, as shown in Fig. 1. The Rayleigh quotient in this case is

$$R = \frac{\int_0^L EI(x)\left[\frac{d^2w(x)}{dx^2}\right]^2 dx + kw^2(L)}{\int_0^L m(x)w^2(x)dx} \tag{21}$$

where $EI(x)$ is the bending stiffness, k is the spring constant, and $m(x)$ is the mass per unit length. The boundary conditions are

$$w = 0 \quad \text{at} \quad x = 0 \tag{22a}$$

$$\frac{dw}{dx} = 0 \text{ at } x = 0 \tag{22b}$$

$$EI\frac{d^2w}{dx^2} = 0 \text{ at } x = L \tag{22c}$$

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = kw \text{ at } x = L \tag{22d}$$

A possible set of admissible functions consists of the cantilever modes, which implies violation of the boundary condition (22d). To satisfy boundary condition (22d), as well as the differential equation in the neighborhood of $x = L$, it is necessary to take an infinity of cantilever modes. Indeed, cantilever modes are characterized by zero shearing force at $x = L$, while boundary condition (22d) requires that the shearing force have some finite value. Hence, a solution in terms of cantilever modes is likely to converge very slowly.

The above problem was addressed in Ref. 8, in which the concept of quasicomparison functions was advanced. Quasicomparison functions were defined in Ref. 8 as linear combinations of admissible functions capable of satisfying all the boundary conditions. A suitable set of quasicomparison functions can be obtained by adding clamped-pinned modes to the clamped-free cantilever mode. In this manner, the possibility is provided for satisfying boundary condition (22d). It should be pointed out that the approximate solution $w^{(n)}(x)$ is not forced in advance to satisfy boundary condition (22d), thus constraining the coefficients a_i. Indeed, the determination of a_i is still left to the process of rendering the Rayleigh quotient stationary, but now it is possible to satisfy boundary condition (21d) with only a relatively small number n of terms. Even though the satisfaction is only approximate, it is greatly improved. In the process, the satisfaction of the differential equation is also improved. It is shown in Ref. 8 that the

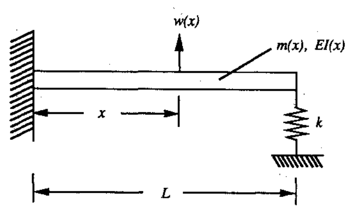

Fig. 1 A cantilever beam in bending supported by a spring.

solution in terms of quasicomparison functions consisting of a combination of several types of admissible functions can vastly improve the convergence properties compared to a solution in terms of admissible functions of a single type.

It is perhaps appropriate to mention here that the slow convergence resulting from a poor choice of admissible functions was observed by other investigators as well, but the explanation and suggested remedy differ markedly from the ones in Ref. 8. In this regard, we single out an investigation by Baruh and Tadikonda,[11] which focuses the attention on the so-called "complementary boundary conditions," defined as quantities in the boundaries that cannot be prespecified. In the case of a bar in axial vibration fixed at one end, the complementary boundary condition consists of the requirement that the first derivative be different from zero at the fixed end. Otherwise, the force is zero at the fixed end, which is in conflict with the physics of the problem. Hence, the modes of a cantilever beam in bending are not suitable as admissible functions for the bar in axial vibration fixed at one end. As it can be concluded from Ref. 8, the class of quasicomparison functions is assumed to satisfy any complementary boundary conditions by definition, although the term "complementary boundary conditions" is never used explicitly. The errors are attributed in Ref. 8 to poor satisfaction of the natural boundary conditions, as well as of the differential equation. The quasicomparison functions were created for the purpose of eliminating these errors.

V. Inclusion Principle for Flexible Multibody Systems Modeled by Rayleigh-Ritz Based Substructure Synthesis

The problem of modeling flexible multibody systems of the type shown in Fig. 2 was discussed extensively in the Introduction. In this paper, we are concerned with the Rayleigh-Ritz based substructure synthesis developed in Ref. 9. It is shown in Ref. 9 that the structure can be envisioned as a collection of substructures consisting of a central substructure and a number of peripheral substructures. The motion of the central substructure consists of rigid-body motions and elastic motions. Then, using a consistent kinematical procedure, the motion of the peripheral substructures is due entirely to elasticity. Assuming small motions, the rigid-body motions of the central substructure can be regarded as consisting of the translational displacement vector $\boldsymbol{R}_0$ and the angular displacement vector $\boldsymbol{\theta}$. Moreover, the elastic displacement vector for all substructures can be written in the form

$$\boldsymbol{w}_s(\boldsymbol{r}_s, t) = \Phi_s(\boldsymbol{r}_s)\boldsymbol{q}_s(t), \; s = 0, a; \; a = 1, 2, \ldots, N \tag{23}$$

where $\boldsymbol{r}_s$ is a vector giving the nominal position in substructure s, Φ_s is a matrix of quasicomparison functions (see Sec. IV) and $\boldsymbol{q}_s(t)$ is a vector of generalized coordinates; N is the number of peripheral substructures. If the dimension of $\boldsymbol{q}_s$ is n_s, then the total number of degrees of freedom is $n = 6 + \Sigma_{s=0}^{N} n_s$. The corresponding configuration vector is $\boldsymbol{x} = [\boldsymbol{R}_0^T \; \boldsymbol{\theta}^T \; \boldsymbol{q}_0^T \; \boldsymbol{q}_1^T \; \boldsymbol{q}_2^T \ldots \boldsymbol{q}_N^T]^T$.

It is shown in Ref. 9 that the free-vibration equations of motion can be written in the compact form

$$M\ddot{\boldsymbol{x}}(t) + K\boldsymbol{x}(t) = 0 \tag{24}$$

where

$$M = \begin{bmatrix} m_t & \tilde{S}_t^T & \bar{\Phi}_t & C_1^T\bar{\Phi}_1 & C_2^T\bar{\Phi}_2, & \ldots, & C_N^T\bar{\Phi}_N \\ \tilde{S}_t & I_t & \tilde{\Phi}_t & H_1 & H_2, & \ldots, & H_N \\ \bar{\Phi}_t^T & \tilde{\Phi}_t^T & M_t & J_1 & J_2, & \ldots, & J_N \\ \bar{\Phi}_1^T C_1 & \tilde{\Phi}_1^T C_1 & J_1^T & M_1 & 0, & \ldots, & 0 \\ \bar{\Phi}_2^T C_2 & \tilde{\Phi}_2^T C_2 & J_2^T & 0 & M_2, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot, & \ldots, & \cdot \\ \bar{\Phi}_N^T C_N & H_N^T & J_N^T & 0 & 0, & \ldots, & M_N \end{bmatrix} \tag{25}$$

is the mass matrix, where

$$m_t = m_o + \sum_{a=1}^{N} m_a, \qquad \tilde{S}_t = \tilde{S}_o + \sum_{a=1}^{N} (m_a\tilde{r}_{oa} + C_a^T\tilde{S}_a C_a)$$

$$\bar{\Phi}_t = \bar{\Phi}_o + \sum_{a=1}^{N} (m_a\Phi_{oa} - C_a^T\tilde{S}_a C_a\Gamma_a)$$

$$I_t = I_o + \sum_{a=1}^{N} (C_a^T I_a C_a - m_a\tilde{r}_{oa}^2 - \tilde{r}_{oa}C_a^T\tilde{S}_a C_a - C_a^T\tilde{S}_a C_a\tilde{r}_{oa})$$

$$\tilde{\Phi}_t = \tilde{\Phi}_o + \sum_{a=1}^{N} [(m_a\tilde{r}_{oa} + C_a^T\tilde{S}_a C_a)\Phi_{oa} + (C_a^T I_a C_a - \tilde{r}_{oa}C_a^T\tilde{S}_s C_a)\Gamma_a]$$

$$M_t = M_o + \sum_{a=1}^{N} (m_a\Phi_{oa}^T\Phi_{oa} - \Phi_{oa}^T C_a^T\tilde{S}_a C_a\Gamma_a + \Gamma_a^T C_a^T\tilde{S}_a C_a\Phi_{oa} + \Gamma_a^T C_a^T I_a C_a\Gamma_a)$$

$$H_s = C_s^T\tilde{\Phi}_s + \tilde{r}_{os}C_s^T\bar{\Phi}_s, \qquad J_s = \Gamma_s^T C_s^T\tilde{\Phi}_s + \Phi_{os}^T C_s^T\bar{\Phi}_s$$

$$M_s = \int_{D_s} \rho_s\Phi_s^T\Phi_s \mathrm{d}D_s$$

$$s = o, a; \qquad a = 1, 2, \ldots, N \tag{26}$$

in which

$$m_s = \int_{D_s} \rho_s \mathrm{d}D_s, \; \tilde{S}_s = \int_{D_s} \rho_s\tilde{r}_s \mathrm{d}D_s, \; I_s = \int_{D_s} \rho_s\tilde{r}_s\tilde{r}_s^T \mathrm{d}D_s$$

$$\bar{\Phi}_s = \int_{D_s} \rho_s\Phi_s \mathrm{d}D_s, \qquad \tilde{\Phi}_s = \int_{D_s} \rho_s\tilde{r}_s\Phi_s \mathrm{d}D_s$$

$$\Phi_{os} = \Phi_o(\boldsymbol{r}_{os}), \qquad \Gamma_s = \nabla \times \Phi_o(\boldsymbol{r}_{os}) \tag{27}$$

Moreover, the stiffness matrix can be written in the form

$$K = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14}^1 & \kappa_{14}^2, & \ldots, & \kappa_{14}^N \\ \kappa_{12}^T & \kappa_{22} & \kappa_{23} & \kappa_{24}^1 & \kappa_{24}^2, & \ldots, & \kappa_{24}^N \\ \kappa_{13}^T & \kappa_{23}^T & K_o + \kappa_{33} & \kappa_{34}^1 & \kappa_{34}^2, & \ldots, & \kappa_{34}^N \\ (\kappa_{14}^1)^T & (\kappa_{24}^1)^T & (\kappa_{34}^1)^T & K_1 + \kappa_{44}^1 & 0, & \ldots, & 0 \\ (\kappa_{14}^2)^T & (\kappa_{24}^2)^T & (\kappa_{34}^2)^T & 0 & K_2 + \kappa_{44}^2, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot, & \ldots, & \cdot \\ (\kappa_{14}^N)^T & (\kappa_{24}^N)^T & (\kappa_{34}^N)^T & 0 & 0, & \ldots, & K_N + \kappa_{44}^N \end{bmatrix} \tag{28}$$

in which $K_o, K_1, \ldots, K_N$ are substructure stiffness matrices and

$$\kappa_{11} = \sum_{a=1}^{N} k_a, \qquad \kappa_{12} = -\sum_{a=1}^{N} k_a(\tilde{r}_{oa} + C_a^T\tilde{r}_{ab}C_a)$$

$$\kappa_{13} = \sum_{a=1}^{N} k_a\Phi_{oa}$$

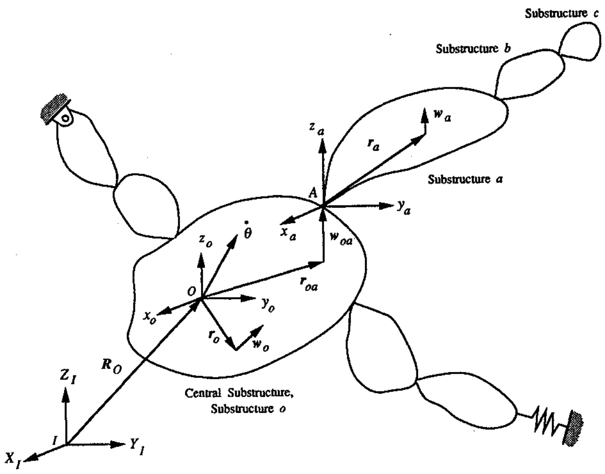

Fig. 2 A flexible multibody system.

$$\kappa_{22} = -\sum_{a=1}^{N} (\tilde{r}_{oa} + C_a^T\tilde{r}_{ab}C_a)k_a(\tilde{r}_{oa} + C_a^T\tilde{r}_{ab}C_a)$$

$$\kappa_{23} = \sum_{a=1}^{N} (\tilde{r}_{oa} + C_a^T\tilde{r}_{ab}C_a)k_a\Phi_{oa}$$

$$\kappa_{33} = \sum_{a=1}^{N} \Phi_{oa}^T k_a \Phi_{oa}, \qquad \kappa_{14}^a = k_a C_a^T \Phi_{ab}$$

$$\kappa_{24}^a = (\tilde{r}_{oa} + C_a^T\tilde{r}_{ab}C_a)k_a C_a^T\Phi_{ab}, \qquad a = 1, 2, \ldots, N$$

$$\kappa_{34}^a = \Phi_{oa}^T k_a C_a^T \Phi_{ab}$$

$$\kappa_{44}^a = \Phi_{ab}^T C_a k_a C_a^T \Phi_{ab}, \qquad a = 1, 2, \ldots, N \tag{29}$$

where

$$\Phi_{ab} = \Phi_a(r_{ab}) \tag{30}$$

Because free vibration of conservative systems is harmonic, $x(t) = e^{i\omega t}x$, Eq. (24) yields the eigenvalue problem

$$Kx = \lambda M x, \quad \lambda = \omega^2 \tag{31}$$

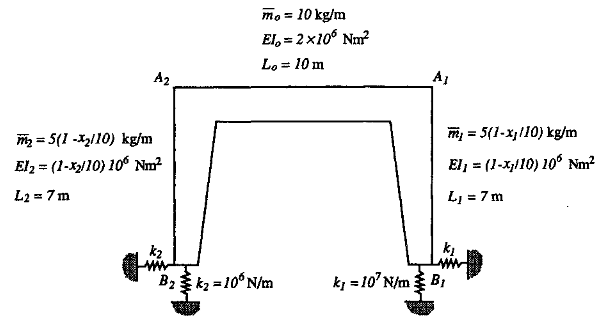

Fig. 3 An elastic frame as a multibody system.

It is not difficult to see that the motion of each substructure is modeled independently of the other. This implies that the number n_s of elastic degrees of freedom for substructure s can be increased one at a time. This results in a new mass and stiffness matrices with an extra row and column and with the previously derived entries remaining unaffected. It follows that the matrices M and K possess the embedding property, so that *the inclusion principle holds for the Rayleigh-Ritz based substructure synthesis.*

VI. Illustrative Example

As an example, we consider the frame shown in Fig. 3, in which the horizontal beam plays the role of the central substructure and the two columns play the role of peripheral substructures. The figure contains a description of the various parameters. This is a planar problem, so that the number of degrees of freedom is $n = 3 + \Sigma_{s=0}^{2} n_s$. The three members are assumed to undergo bending vibration in the transverse direction only. The quasicomparison functions for the central substructure consist of linear combinations of pinned-pinned functions and cosine functions, and those for the peripheral substructures consist of linear combinations of clamped-free and clamped-pinned functions. The functions are displayed in Fig. 4. The total number of degrees of freedom and the number of elastic degrees of freedom of each substructure are shown in Table 1.

To verify the embedding property, we use eight terms in series (8) and obtain the mass and stiffness matrices

$$M^{(8)} = \begin{bmatrix} 145.500 & 0.000 & 130.667 & 0.000 & 0.000 & 13.468 & 18.341 & 13.468 \\ 0.000 & 145.500 & 0.000 & 63.662 & 0.000 & 0.000 & 0.000 & 0.000 \\ 130.667 & 0.000 & 2513.917 & 0.000 & 430.142 & 62.979 & 65.764 & 62.979 \\ 0.000 & 63.662 & 0.000 & 103.600 & 0.000 & -19.785 & -20.660 & 19.785 \\ 0.000 & 0.000 & 430.142 & 0.000 & 95.500 & 0.000 & 0.000 & 0.000 \\ 13.468 & 0.000 & 62.979 & -19.785 & 0.000 & 15.240 & 13.815 & 0.000 \\ 18.341 & 0.000 & 65.764 & -20.660 & 0.000 & 13.815 & 21.070 & 0.000 \\ 13,468 & 0.000 & 62.979 & 19.785 & 0.000 & 0.000 & 0.000 & 15.240 \end{bmatrix} \tag{32a}$$

$$K^{(8)} = \begin{bmatrix} 1.100 & 0.000 & 7.700 & -1.979 & 0.000 & 2.000 & 0.000 & 0.200 \\ 0.000 & 1.100 & 4.500 & 0.000 & 0.900 & 0.000 & 0.000 & 0.000 \\ 7.700 & 4.500 & 81.400 & -13.854 & 5.500 & 14.000 & 0.000 & 1.400 \\ -1.979 & 0.000 & -13.854 & 5.329 & 0.000 & -4.398 & 0.000 & 0.440 \\ 0.000 & 0.900 & 5.500 & 0.000 & 1.110 & 0.000 & 0.000 & 0.000 \\ 2.000 & 0.000 & 14.000 & -4.398 & 0.000 & 4.003 & 0.004 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.004 & 0.048 & 0.000 \\ 0.200 & 0.000 & 1.400 & 0.440 & 0.000 & 0.000 & 0.000 & 0.403 \end{bmatrix} \times 10^7 \tag{32b}$$

The lower triangular matrix resulting from the Cholesky decomposition of $M^{(8)}$ is

$$L^{(8)} = \begin{bmatrix} 12.062 & & & & & & & \\ 0.000 & 12.062 & & & & 0 & & \\ 10.833 & 0.000 & 48.955 & & & & & \\ 0.000 & 5.278 & 0.000 & 8.703 & & & & \\ 0.000 & 0.000 & 8.787 & 0.000 & 4.278 & & & \\ 1.117 & 0.000 & 1.039 & -2.273 & -2.135 & 1.785 & & \\ 1.520 & 0.000 & 1.007 & -2.374 & -2.068 & 0.705 & 2.708 & \\ 1.117 & 0.000 & 1.039 & 2.273 & -2.135 & -0.962 & -0.401 & 1.449 \end{bmatrix} \tag{33}$$

Then, using Eq. (16), we obtain the single symmetric matrix

$$A^{(8)} = \begin{bmatrix} 7.560 & 0.000 & 11.367 & -18.853 & -23.349 & 29.602 & -50.536 & -1.682 \\ 0.000 & 7.560 & 7.621 & -4.585 & 1.789 & -8.136 & -3.367 & -1.970 \\ 11.367 & 7.621 & 28.565 & -32.967 & -32.410 & 35.162 & -79.807 & -6.803 \\ -18.853 & -4.585 & -32.967 & 73.140 & 57.140 & -90.628 & 154.189 & 24.962 \\ -23.349 & 1.789 & 32.410 & 57.140 & 73.275 & -92.980 & 155.415 & 5.466 \\ 29.602 & -8.136 & 35.162 & -90.628 & -92.980 & 147.625 & -217.774 & -20.082 \\ -50.536 & -3.367 & -79.807 & 154.189 & 155.415 & -217.774 & 374.980 & 37.059 \\ -1.682 & -1.970 & -6.803 & 24.962 & 5.466 & -20.082 & 37.059 & 24.574 \end{bmatrix} \times 10^4 \tag{34}$$

defining the eigenvalue problem in standard form, Eq. (15). Note that, according to Table 1, the motion of the frame is represented by two rigid-body translations, one rigid-body rotation, two quasicomparison functions for the horizontal beam, two quasicomparison functions for the right column, and one quasicomparison function for the left column. Similarly, for $n = 9$, we obtain the mass and stiffness matrices

$$M^{(9)} = \begin{bmatrix} 145.500 & 0.000 & 130.667 & 0.000 & 0.000 & 13.468 & 18.341 & 13.468 & 18.341 \\ 0.000 & 145.500 & 0.000 & 63.662 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 130.667 & 0.000 & 2513.917 & 0.000 & 430.142 & 62.979 & 65.764 & 62.979 & 65.764 \\ 0.000 & 63.662 & 0.000 & 103.600 & 0.000 & -19.785 & -20.660 & 19.785 & 20.660 \\ 0.000 & 0.000 & 430.142 & 0.000 & 95.500 & 0.000 & 0.000 & 0.000 & 0.000 \\ 13.468 & 0.000 & 62.979 & -19.785 & 0.000 & 15.240 & 13.815 & 0.000 & 0.000 \\ 18.341 & 0.000 & 65.764 & -20.660 & 0.000 & 13.815 & 21.070 & 0.000 & 0.000 \\ 13.468 & 0.000 & 62.979 & 19.785 & 0.000 & 0.000 & 0.000 & 15.240 & 13.815 \\ 18.341 & 0.000 & 65.764 & 20.660 & 0.000 & 0.000 & 0.000 & 13.815 & 21.070 \end{bmatrix} \tag{35a}$$

$$K^{(9)} = \begin{bmatrix} 1.100 & 0.000 & 7.700 & -1.979 & 0.000 & 2.000 & 0.000 & 0.200 & 0.000 \\ 0.000 & 1.100 & 4.500 & 0.000 & 0.900 & 0.000 & 0.000 & 0.000 & 0.000 \\ 7.700 & 4.500 & 81.400 & -13.854 & 5.500 & 14.000 & 0.000 & 1.400 & 0.000 \\ -1.979 & 0.000 & -13.854 & 5.329 & 0.000 & -4.398 & 0.000 & 0.440 & 0.000 \\ 0.000 & 0.900 & 5.500 & 0.000 & 1.110 & 0.000 & 0.000 & 0.000 & 0.000 \\ 2.000 & 0.000 & 14.000 & -4.398 & 0.000 & 4.003 & 0.004 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.004 & 0.048 & 0.000 & 0.000 \\ 0.200 & 0.000 & 1.400 & 0.440 & 0.000 & 0.000 & 0.000 & 0.403 & 0.004 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.004 & 0.048 \end{bmatrix} \times 10^7 \tag{35b}$$

the lower triangular matrix

$$L^{(9)} = \begin{bmatrix} 12.062 & & & & & & & & \\ 0.000 & 12.062 & & & & 0 & & & \\ 10.833 & 0.000 & 48.955 & & & & & & \\ 0.000 & 5.278 & 0.000 & 8.703 & & & & & \\ 0.000 & 0.000 & 8.787 & 0.000 & 4.278 & & & & \\ 1.117 & 0.000 & 1.039 & -2.273 & -2.135 & 1.785 & & & \\ 1.520 & 0.000 & 1.007 & -2.374 & -2.068 & 0.705 & 2.708 & & \\ 1.117 & 0.000 & 1.039 & 2.273 & -2.135 & -0.962 & -0.401 & 1.449 & \\ 1.520 & 0.000 & 1.007 & 2.374 & -2.068 & -0.988 & -0.470 & 0.083 & 2.574 \end{bmatrix} \tag{36}$$

and the single symmetric matrix

$$A^{(9)} = \begin{bmatrix} 7.560 & 0.000 & 11.367 & -18.853 & -23.349 & 29.602 & -50.536 & -1.682 & -8.089 \\ 0.000 & 7.560 & 7.621 & -4.585 & 1.789 & -8.136 & -3.367 & -1.970 & -0.989 \\ 11.367 & 7.621 & 28.565 & -32.967 & -32.410 & 35.162 & -79.807 & -6.803 & -14.371 \\ -18.853 & -4.585 & -32.967 & 73.140 & 57.140 & -90.628 & 154.189 & 24.962 & -4.962 \\ -23.349 & 1.789 & -32.410 & 57.140 & 73.275 & -92.980 & 155.415 & 5.466 & 25.139 \\ 29.602 & -8.136 & 35.162 & -90.628 & -92.980 & 147.625 & -217.774 & -20.082 & -4.795 \\ -50.536 & -3.367 & -79.807 & 154.189 & 155.415 & -217.774 & 374.980 & 37.059 & 27.384 \\ -1.682 & -1.970 & -6.803 & 29.962 & 5.466 & -20.082 & 37.059 & 24.574 & -15.661 \\ -8.089 & -0.989 & -14.371 & -4.962 & 25.139 & -4.795 & 27.384 & -15.661 & 46.094 \end{bmatrix} \times 10^4 \tag{37}$$

We note that the case $n = 9$ is obtained from the case $n = 8$ by adding one quasicomparison function to the left column of the frame. It is easy to verify by comparing Eqs. (34) and (37) that the coefficient matrices $A^{(n)}$ possess the embedding property, so that the computed eigenvalues satisfy the inclusion principle.

The eigenvalue problem was solved 25 times for $n = 6, 7, \ldots, 30$. The computed natural frequencies measured in Hertz are given in Table 2. It is easy to verify that the results corroborate the theory developed here. In particular, the results demonstrate the uniform convergence of the computed eigenvalues. This convergence is displayed in the plots of $\varepsilon_r^{(n)}$ vs n of Fig. 5, where

$$\varepsilon_r^{(n)} = \frac{\omega_r^{(n)} - \omega_r}{\omega_r}, \quad r = 1, 2 \tag{38}$$

is the normalized error in the rth natural frequency and n is the number of terms in the approximating series. Note that, for plotting purposes, we used $\omega_r = \omega_r^{(30)}$. Also note that the error is plotted in Fig. 5 using a logarithmic scale, so that the convergence is even more dramatic than can be inferred from

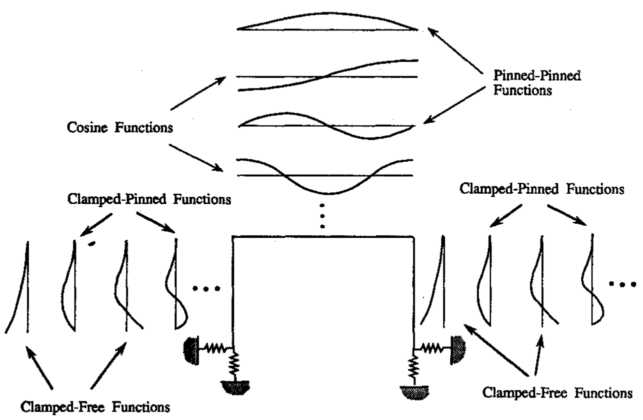

Fig. 4 Quasicomparison functions for the elastic frame.

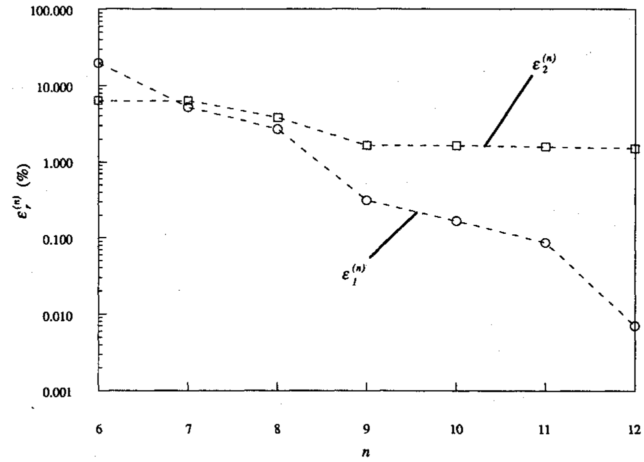

Fig. 5 Normalized error in computed natural frequencies vs the number of terms in the approximating series.

Table 1 Number of degrees of freedom

n_o	n_1	n_2	n
1	1	1	6
2	1	1	7
2	2	1	8
2	2	2	9
3	2	2	10
⋮	⋮	⋮	⋮
9	9	9	30

Table 2 Computed natural frequencies (H_z)

n	ω_r^n					
6	1.75745	8.71827	26.92279	81.05934	85.64478	264.00941
7	1.54617	8.71674	24.63322	33.86527	85.27040	89.04972
	276.08209					
8	1.51043	8.51557	21.09261	27.66880	38.04814	86.35866
	95.37749	409.22782				
9	1.47530	8.33865	20.24966	22.77297	28.49704	48.43519
	93.38683	133.00602	409.70813			
10	1.47317	8.33860	20.09471	22.62309	27.60450	39.76323
	90.56847	120.33963	155.10804	419.68652		
11	1.47200	8.33338	20.00337	22.59772	27.32456	39.69343
	71.76911	91.01342	124.51966	173.49633	570.02728	
12	1.47084	8.32848	19.93381	22.42968	27.30103	39.18431
	68.41797	72.61711	91.29930	172.81259	208.98397	571.27570
⋮	⋮	⋮	⋮	⋮	⋮	⋮
29	1.47073	8.20443	19.93101	21.12023	25.66975	38.34873
	59.77561	68.94368	75.10177	96.49401	121.31098	133.58491
	149.47475	189.38001	210.41994	244.71701	285.39690	315.03814
	366.36635	401.81190	448.56316	526.29984	639.17033	678.97218
	900.56616	1157.16696	1349.49227	1521.09589	1860.84123	
30	1.47073	8.20443	19.93100	21.12022	25.66975	38.34872
	59.77560	68.94354	75.10164	96.49382	121.31090	133.58434
	149.47320	189.37598	210.41788	244.71458	285.35671	315.02416
	366.22456	401.56686	448.47038	526.29657	602.39638	678.24583
	760.77295	901.45745	1216.92502	1406.27543	1591.73991	1862.27319

the figure. The same trend holds true for the higher computed natural frequencies, as can be concluded from Table 2.

VII. Summary and Conclusions

Over the last three decades, a number of procedures for modeling complex structures have been developed. In the case in which a structure consists of an aggregation of identifiable substructures, it is possible to model each substructure separately and then force the individual substructures to act as a single structure by imposing appropriate compatibility constraints. This is the essence of the component-synthesis method. Since the original paper by Hurty, a host of variants have been advanced, all differing in the type of "component modes" used. Many of these procedures suffer from a drawback, in that there are no good ways of proving convergence.

Quite recently, a substructure synthesis has been developed that has many of the advantages of the Rayleigh-Ritz method and at the same time avoids problems of slow convergence. This substructure synthesis derives its good characteristics from the representation of substructures motion by means of quasi-comparison functions. Among these advantages is the one that the mass and stiffness matrices possess the embedding property. As a result, the inclusion principle exists, so that the eigenvalues computed using the Rayleigh-Ritz based substructure synthesis converge to the actual eigenvalues; the convergence is uniform and from above.

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